MATH 2050 - The Completeness Property of $\mathbb{R}$

(Reference: Bartle §2.3)

Def$^{\underline{n}}$ / Thm (Completeness Property of $\mathbb{R}$)

Every $\phi \neq S \subseteq \mathbb{R}$ that has an "upper bound" ?

must have a "supremum" in $\mathbb{R}$.
?

We first make sense of the ?'s.

Def$^{\underline{n}}$: Let $\phi \neq S \subseteq \mathbb{R}$.

(a) S is **bounded above** if $\exists\, u \in \mathbb{R}$ s.t. $s \leq u \quad \forall\, s \in S$

Any such $u \in \mathbb{R}$ is called an **upper bound of S**.

(b) S is **bounded below** if $\exists\, w \in \mathbb{R}$ s.t. $s \geq w \quad \forall\, s \in S$

Any such $w \in \mathbb{R}$ is called a **lower bound of S**.

(c) S is **bounded** if it is both bdd above AND below.

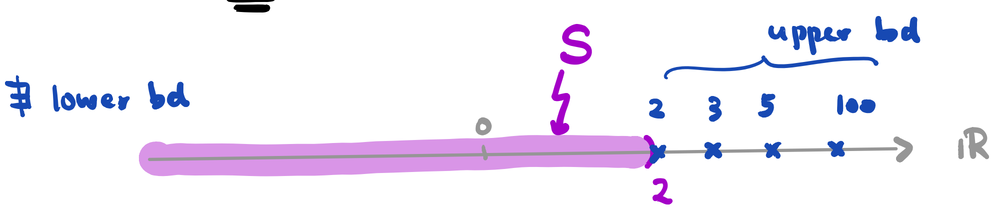
Otherwise, S is **unbounded**.

Example: $S := \{ x \in \mathbb{R} \mid x < 2 \}$

Note: There are many upper bds, e.g. $2, 3, 5, 100, \sqrt{100}$ etc...

$\Rightarrow$ S is bdd above.

But S is NOT bdd below. (Ex: prove it)



Def$^{\underline{n}}$: Let $\phi \neq S \subseteq \mathbb{R}$.

(a) Suppose S is bdd above.

Then, $u \in \mathbb{R}$ is called a supremum (or least upper bound) of S if the following holds:

[Notation: $u = \sup S$ or $\text{l.u.b.} S$]

(i) u is an upper bd of S

(ii) $u \leq v$ for any upper bd v of S

(b) Similarly, we can define infimum (or greatest lower bound)

[Notation: $\inf S$ or $\text{g.l.b.} S$]   Ex: Write this down.

Lemma: $\sup S$, if exists, is unique.

Proof: Suppose there are two $u, w \in \mathbb{R}$ which are supremum of S

Therefore, u, w satisfy (i), (ii) in the def$^{\underline{n}}$ above.

By (i) for w and (ii) for u, we have

$$u \leq w \quad \because w \text{ is an upper bd}$$

Similarly, by (i) for u and (ii) for w, we have

$$w \leq u \quad \because u \text{ is an upper bd.}$$

Thus, $u = w$. $\square$

We now establish a useful way to prove that $u \in \mathbb{R}$ is the supremum of a subset $S \subseteq \mathbb{R}$.

Prop: Let $\phi \neq S \subseteq \mathbb{R}$. Then $u = \sup S$ iff

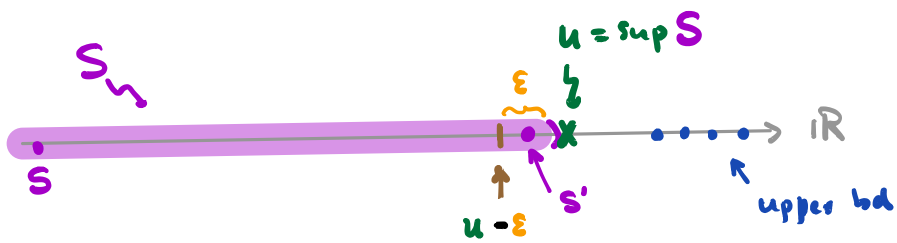
(i) u is an upper bound of S, i.e.

$$\boxed{s \leq u \qquad \forall\, s \in S}$$

(ii) u is the smallest upper bound of S, i.e.

$$\boxed{\forall\, \varepsilon > 0,\ \exists\, s' \in S \text{ s.t. } u - \varepsilon < s'}$$

Picture:



Proof: "$\Rightarrow$" Suppose $u = \sup S$.

By (i), u is an upper bd of S

$\Rightarrow \quad u \geq s \qquad \forall\, s \in S$ which is (i).

By (ii), $u \leq v$ for any upper bd. v of S. (*)

Fix $\varepsilon > 0$, but arbitrary.

Since $u - \varepsilon < u$, (*) $\Rightarrow$ $u - \varepsilon$ cannot be an upper bd.

So, $\exists\, s' \in S$ s.t. $u - \varepsilon < s'$

"$\Leftarrow$" Exercise. $\square$

Similarly, for infimum, we have:

Prop: $u = \inf S$ iff

① $s \geq u$, $\forall s \in S$ (i.e. u is a lower bd)

② $\forall \varepsilon > 0$, $\exists s' \in S$ s.t. $u + \varepsilon > s'$ (i.e. u is the greatest lower bd.)

Pf: Exercise.

Examples:

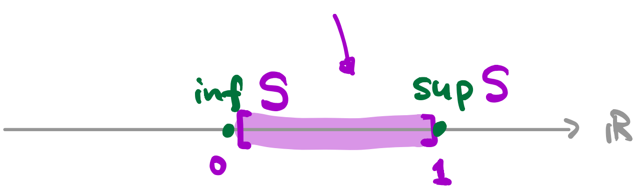
1) $S = \{s_1, \ldots, s_n\}$ "finite set" (Assume: $s_1 < s_2 < \cdots < s_n$)

$\sup S = s_n$

$\inf S = s_1$

(Ex: Prove this)

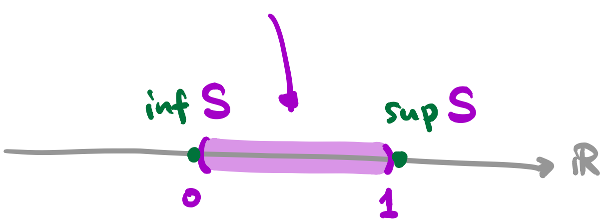
2) $S = [0, 1]$



$\sup S = 1 \in S$

$\inf S = 0 \in S$

3) $S = (0, 1)$



$\sup S = 1 \notin S$

$\inf S = 0 \notin S$

Recall that:

Completeness Property: Every $\phi \neq S \subseteq \mathbb{R}$ which is bounded above must have a supremum in $\mathbb{R}$. [Note: $\mathbb{Q}$ fails this!]

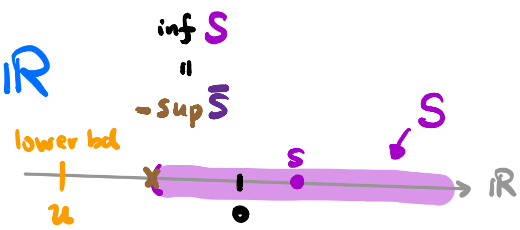
Q: What about the existence of infimum?

A: It follows from the completeness property.

Prop: Every $\phi \neq S \subseteq \mathbb{R}$ that is bounded below must have an infimum in $\mathbb{R}$.

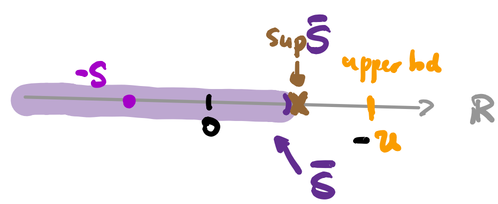
Proof: Given $\phi \neq S \subseteq \mathbb{R}$, consider the subset:

$$\phi \neq \bar{S} := \{ -s \mid s \in S \} \subseteq \mathbb{R}$$



Claim: $\bar{S}$ is bdd above.

Pf of Claim:



Since S is bdd below, i.e.

$\exists$ some lower bd. u of S

$\Leftrightarrow \quad u \leq s \quad \forall s \in S$

$\Rightarrow \quad -u \geq -s \quad \forall s \in S$

$\Rightarrow \quad -u$ is an upper bd for $\bar{S}$ ie. $\bar{S}$ is bdd above.

By Completeness Property, $\sup \bar{S}$ exists in $\mathbb{R}$.

Claim: $\inf S$ exists. $\inf S = -\sup \bar{S}$.

Pf of Claim:

Check: $-\sup \bar{S}$ is a lower bd for S (Ex:)

This is the same by reversing the arguments of the claim above.

Check: $-\sup \bar{S}$ is the greatest lower bd. for S

Let $\varepsilon > 0$ be fixed but arbitrary.

(Want to show: $\exists\ s' \in S$ s.t. $-\sup \bar{S} + \varepsilon > s'$) — (*)

By ② of supremum for $\bar{S}$,

$$\sup \bar{S} - \varepsilon < \bar{s}' \quad \text{for some } \bar{s}' \in \bar{S}$$

By def^{n}, we write $\bar{s}' = -s'$ for some $s' \in S$

So. $\sup \bar{S} - \varepsilon < -s' \Rightarrow -\sup \bar{S} + \varepsilon > s'$ for some $s' \in S$

which is (*) $\qquad \blacksquare$

Archimedean Property: $\mathbb{N}$ is NOT bdd above.

Pf: Suppose NOT, i.e. $\mathbb{N}$ is bdd above.

By Completeness Property, $\sup \mathbb{N} =: u \in \mathbb{R}$ exists.

So, $u - 1 < n'$ for some $n' \in \mathbb{N}$.

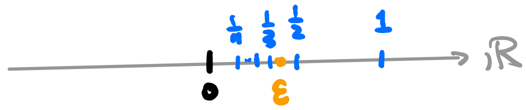
$\Rightarrow \quad u < n' + 1 \in \mathbb{N}$

$\Rightarrow \quad u$ is NOT an upper bd for $\mathbb{N}$ ← Contradiction!

$\blacksquare$

Corollaries:

(i) $\inf \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = 0$



(ii) $\forall\, \varepsilon > 0,\ \exists\, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \varepsilon$

(iii) $\forall\, y > 0,\ \exists!\, n \in \mathbb{N}$ st. $n - 1 < y \leq n$

unique

Ex: Prove these!